Bipartite Matching

Introduction to Computational Mathematics

Instructor: **Tasuku Soma**

July 30, 2026

Outline

1. Min-max theorem for bipartite matching

- Integrality of LP (recap)
- Kőnig–Egerváry theorem

2. Unweighted bipartite matching

- Augmenting-path algorithm
- Revisiting the Kőnig–Egerváry theorem

3. Weighted bipartite matching

- Optimality condition
- Hungarian method

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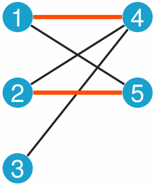
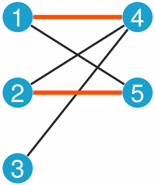
Bipartite matching

Weighted bipartite matching problem

Input $G = (V; E)$: a bipartite graph,
edge weights w_e $(e \in E)$
Output A maximum-weight matching M of G



Bipartite matching



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IP formulation

$$
\begin{aligned}
\text{maximize} \quad & \sum_{e \in E} w_e x_e \\
\text{subject to} \quad & \sum_{e \in \delta(i)} x_e \leq 1 \quad (i \in V) \\
& x_e \in \{0, 1\} \quad (e \in E)
\end{aligned}
$$

$\delta(i)$: the set of edges incident to vertex i

$$
\begin{aligned}
\max \quad & w^\top x \\
\text{s.t.} \quad & x_{14} + x_{15} && \leq 1 \\
& x_{24} + x_{25} && \leq 1 \\
& x_{34} && \leq 1 \\
& x_{14} + x_{24} + x_{34} && \leq 1 \\
& x_{15} + x_{25} && \leq 1 \\
& x_{14}, x_{15}, x_{24}, x_{25}, x_{34} && \in \{0, 1\}
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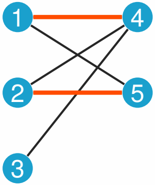


LP formulation

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} w_e x_e \\ \text{subject to} \quad & \sum_{e \in \delta(i)} x_e \le 1 \qquad (i \in V) \\ & x_e \ge 0 \qquad\qquad\quad (e \in E) \end{aligned}$$

$$\begin{aligned} \max \quad & w^\top x \\ \text{s.t.} \quad & x_{14} + x_{15} && \le 1 \\ & x_{24} + x_{25} && \le 1 \\ & x_{34} && \le 1 \\ & x_{14} + x_{24} + x_{34} && \le 1 \\ & x_{15} + x_{25} && \le 1 \\ & x_{14}, x_{15}, x_{24}, x_{25}, x_{34} && \ge 0 \end{aligned}$$

Recap: Total unimodularity of the coefficient matrix ensures an LP optimal solution with **0–1** entries!

Dual problem for bipartite matching

Primal (P)

$$\begin{aligned} \max \quad & \sum_{e \in E} w_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(i)} x_e \leq 1 \quad (i \in V) \\ & x \geq 0 \end{aligned}$$

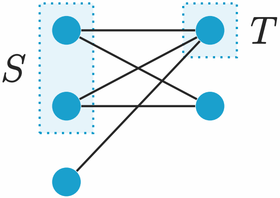
Dual (D)

$$\begin{aligned} \min \quad & \sum_{i \in V} y_i \\ \text{s.t.} \quad & y_i + y_j \geq w_e \quad (e = ij \in E) \\ & y \geq 0 \end{aligned}$$

The coefficient matrix of the dual problem (D) is also totally unimodular. Hence, if edge weights w_e $(e \in E)$ are integers, then (D) has an integral optimal solution.

Kőnig–Egerváry theorem

$G = (V^+, V^-; E)$: bipartite graph with vertex sets V^+ and V^-.



Definition

$S \subseteq V^+, T \subseteq V^-$ is a **vertex cover**

$\overset{\text{def}}{\iff}$ for any edge $e = ij \in E$, either $i \in S$ or $j \in T$.

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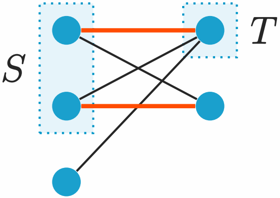

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$\overset{\text{def}}{\iff}$ for any edge $e = ij \in E$, either $i \in S$ or $j \in T$.

Lemma (Weak Duality)

For any matching M and vertex cover (S, T), we have $|M| \leq |S| + |T|$.

(pf) Since M consists of disjoint edges, at least $|M|$ vertices are needed to cover M. $\square$

Kőnig–Egerváry theorem

Consider the previous LP with $w \equiv 1$. By integrality and strong duality, we obtain the following theorem.

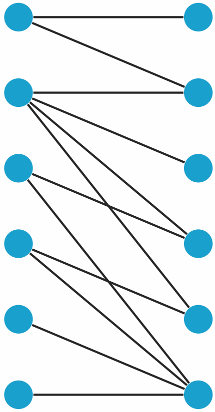
Theorem (Kőnig–Egerváry)

For any bipartite graph G, the following min-max relation holds:

$$\max_{M:\ \textit{matching}} |M| = \min_{(S,T):\ \textit{vertex cover}} |S| + |T|$$

Example

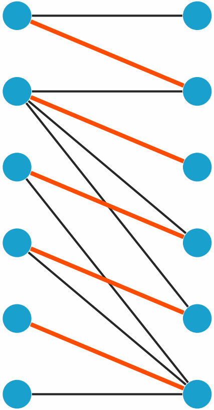
In the bipartite graph on the right,



Example

In the bipartite graph on the right,

- the maximum matching size is $|M| = 5$,



Example

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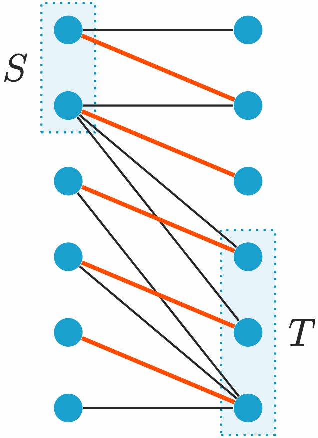
- the maximum matching size is $|M| = 5$,
- the minimum vertex-cover size is $|S| + |T| = 5$.



In general, to prove that a matching M is maximum in a bipartite graph, it suffices to present a vertex cover (S, T) satisfying

$$|M| = |S| + |T|.$$

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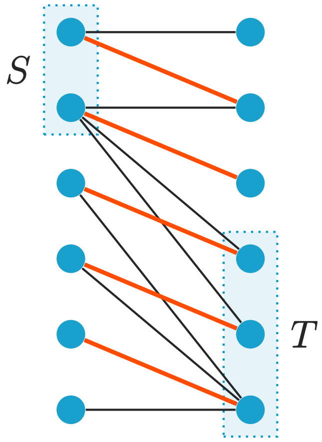

In general, to prove that a matching M is maximum in a bipartite graph, it suffices to present a vertex cover (S, T) satisfying

$$|M| = |S| + |T|.$$

Such a vertex cover is a certificate of optimality of M **(good characterization)**

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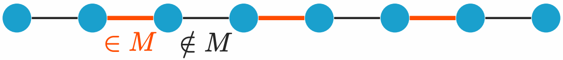
Augmenting paths

M: matching of G

Definition

A path P is an **augmenting path** (with respect to M) $\overset{\text{def}}{\Longleftrightarrow}$

1. edges in $E \setminus M$ and in M appear alternately in P.
2. the two endpoints of P are unmatched in M.



$|M| = 3$

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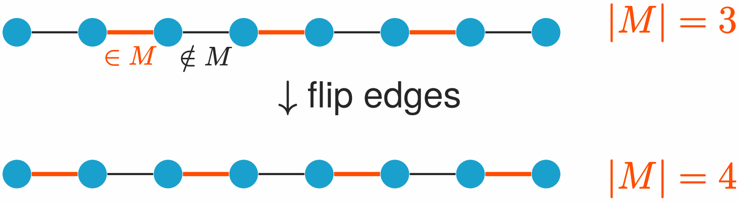

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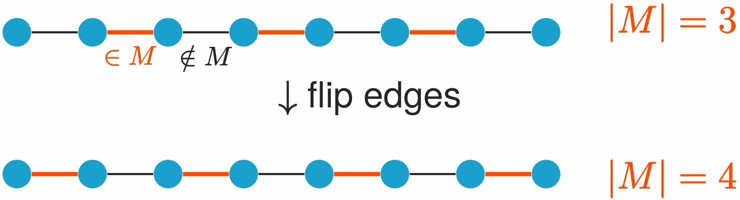


Therefore, if M is maximum, then no augmenting path exists.

Augmenting paths

Lemma

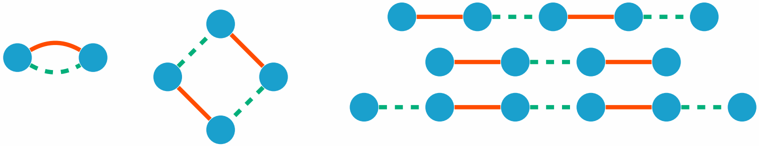
If M is not maximum, then an augmenting path exists.

Augmenting paths

Lemma

If M is not maximum, then an augmenting path exists.

(pf) Take two matchings M, M' with $|M| < |M'|$. Then, each connected component of $M + M'$ is either an alternating path or an even cycle.



Since $|M| < |M'|$, some connected component of $M + M'$ has more M' edges than M edges. This is an augmenting path for M. $\square$

Augmenting-path algorithm

The lemma implies the following algorithm.

Augmenting-path algorithm

Let $M \leftarrow \emptyset$.
while there exists an augmenting path with respect to M :
    Find an augmenting path P.
    Flip edges along P to obtain a larger matching M.
return M

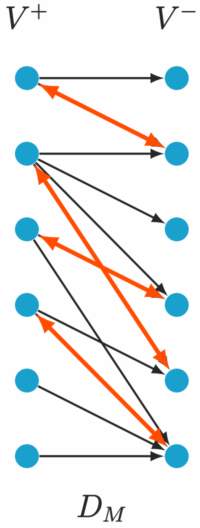
If an augmenting path can be found in A time, then the whole algorithm runs in $O(nA)$ time. $(n = |V|)$

Finding augmenting paths via a directed graph

For a matching M, orient edges of G as follows:

- edges in M: both directions,
- edges in $E \setminus M$: from V^+ to V^-.

Let the resulting digraph be D_M.



Finding augmenting paths via a directed graph

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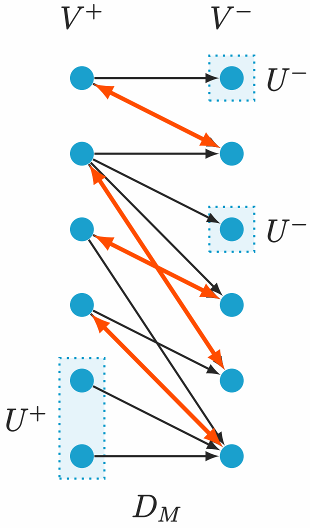
- $U^+ :=$ unmatched vertices in V^+ under M
- $U^- :=$ unmatched vertices in V^- under M



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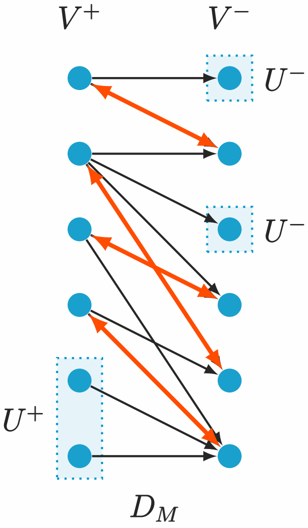
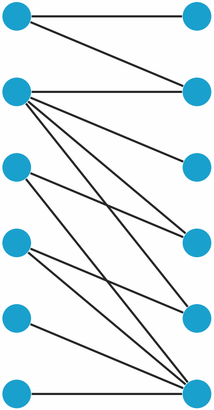
augmenting path $\longleftrightarrow$ directed path from U^+ to U^- in D_M

Hence, BFS finds an augmenting path in $O(m)$ time.

$(m = |E|)$

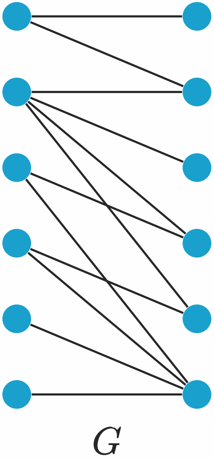


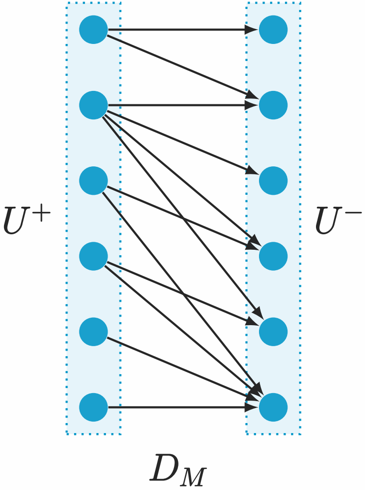
Example



G

Example





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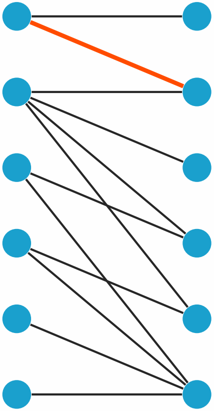

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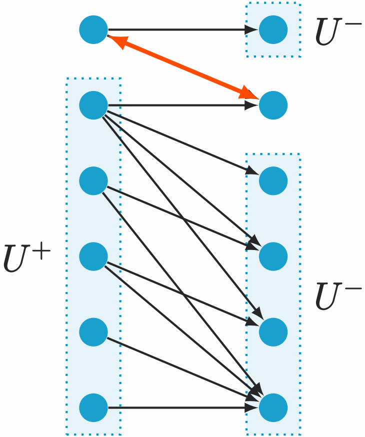
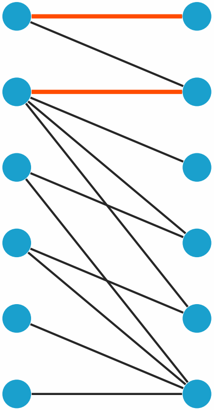


D_M

Example



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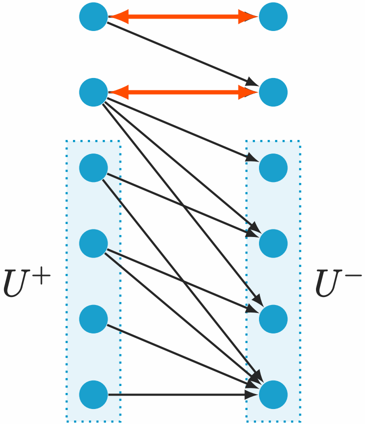

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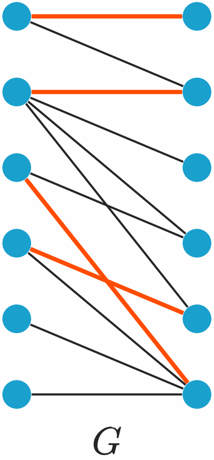

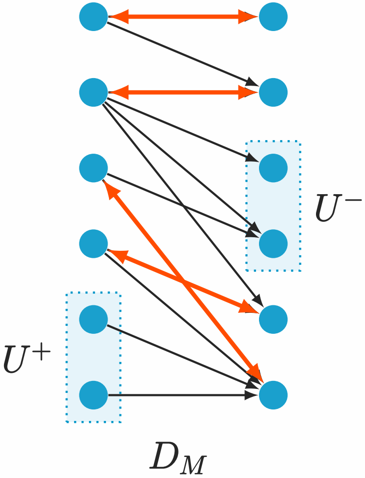

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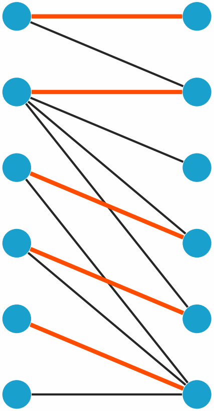

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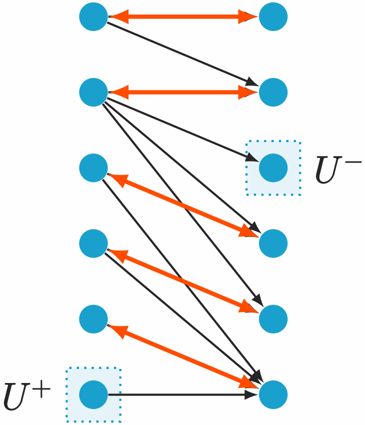


D_M

[Finish]

Revisiting Kőnig–Egerváry

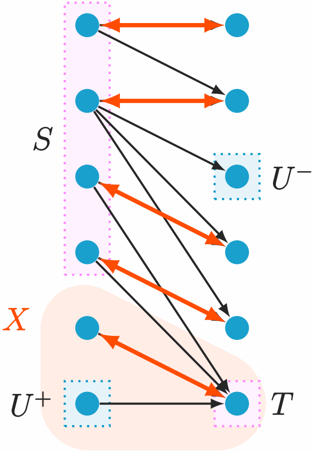
$$\max_{M:\text{matching}} |M| = \min_{(S,T):\text{ vertex cover}} |S| + |T|$$

Theorem

At the termination of the algorithm, let X be the set of all vertices reachable from U^+ in D_M. Define

$$S := V^+ \setminus X, \quad T := V^- \cap X.$$

Then (S, T) is a minimum vertex cover.



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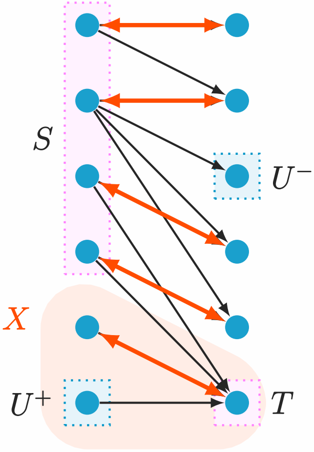

(pf) Why vertex cover? One can traverse any edge from left to right in D_M, so (S,T) is a vertex cover by definition.

Why minimum? Matching edges are bidirected, so their endpoints are either both in X or both not in X. Since no augmenting path exists, $X \cap U^- = \emptyset$. Hence every vertex in (S,T) covers exactly one edge of M. Thus, $|S| + |T| = |M|$. □

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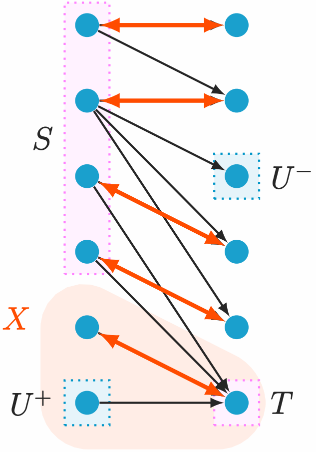

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An algorithmic proof of the Kőnig–Egerváry theorem without LP!

Unweighted bipartite matching (summary)

Theorem

In a bipartite graph:

- *The augmenting-path algorithm finds a maximum matching M in $O(mn)$ time.* $(m = |E|, n = |V|)$
- *It also finds a vertex cover (S, T) satisfying $|M| = |S| + |T|$.*

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Weighted bipartite (perfect) matching

A matching M is **perfect** $\overset{\text{def}}{\iff}$ every vertex is incident to M $\iff$ $2|M| = |V|$.

Weighted perfect bipartite matching problem

Input $G = (V; E)$: bipartite graph ($|V^+| = |V^-| = n/2$), edge weights w_e ($e \in E$)
Output A perfect matching M maximizing $w(M) := \sum_{e \in M} w_e$

Assume that G has at least one perfect matching in what follows.

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Primal (P)

$$\begin{aligned} \max \quad & \sum_{e \in E} w_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(i)} x_e = 1 \quad (i \in V) \\ & x \geq 0 \end{aligned}$$

Dual (D)

$$\begin{aligned} \min \quad & \sum_{i \in V} y_i =: y(V) \\ \text{s.t.} \quad & y_i + y_j \geq w_e \quad (e = ij \in E) \end{aligned}$$

Optimality Condition

Lemma (Optimality condition)

$M \subseteq E$, $y \in \mathbb{R}^V$ are optimal solutions

$\iff$

1. *M is a perfect matching,*
2. *y is feasible for (D),*
3. $y_i + y_j = w_e \quad (e = ij \in M)$

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$\iff$

1. *M is a perfect matching,*
2. *y is feasible for (D),*
3. $y_i + y_j = w_e \quad (e = ij \in M)$

- a feasible solution of (D) is called a **w-cover**,
- edges satisfying condition (3) are said to be **tight** (w.r.t. y).

Primal (P)

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Hungarian method

The algorithm maintains a matching M and a w-cover y satisfying (2)(3), and updates them until (1) is satisfied.

Definition

For a w-cover y, define a subgraph $G_y = (V^+, V^-, E_y)$ of G by

$$E_y = \{e = ij \in E : y_i + y_j = w_e\}$$

That is, G_y keeps only tight edges with respect to y.

Lemma

*If G_y has a **perfect** matching, then it is a maximum-weight matching.*

(pf) Optimality condition! $\square$

Hungarian method

What if G_y has no perfect matching?
$\longrightarrow$ We can update y and improve the objective value of (D).

Hungarian method

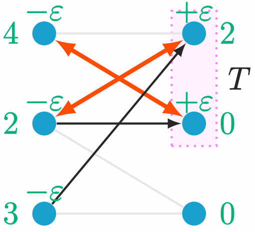
What if G_y has no perfect matching?
$\longrightarrow$ We can update y and improve the objective value of (D).

Lemma

Suppose G_y has no perfect matching. Let (S, T) be a minimum vertex cover of G_y. Define $\varepsilon := \min\{y_i + y_j - w_e : e = ij \in E, i \notin S, j \notin T\}$. Let y' be

$$y'_i = \begin{cases} y_i - \varepsilon & (i \in V^+ \setminus S) \\ y_i + \varepsilon & (i \in V^- \cap T) \\ y_i & (\textit{otherwise}) \end{cases}$$

Then y' is a w-cover with $y'(V) < y(V)$.



$S = \emptyset$

Proof

$$\varepsilon := \min\{y_i + y_j - w_e : e = ij \in E, i \notin S, j \notin T\}.$$
$$y'_i = \begin{cases} y_i - \varepsilon & (i \in V^+ \setminus S) \\ y_i + \varepsilon & (i \in V^- \cap T) \\ y_i & (\text{otherwise}) \end{cases}$$

Why y' is a w-cover?

- The only constraints $y_i + y_j \geq w_e$ ($e = ij \in E$) that could become violated after the update are those with $i \notin S$, $j \notin T$.
- By the definition of ε, feasibility is preserved.

Proof

$$\varepsilon := \min\{y_i + y_j - w_e : e = ij \in E, i \notin S, j \notin T\}.$$

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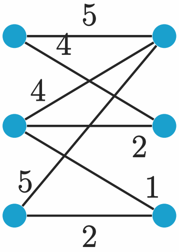
Why $y'(V) < y(V)$?

- $y'(V) = y(V) - \varepsilon(|V^+ \setminus S| - |T|) = y(V) - \varepsilon(n/2 - |S| - |T|)$.
- Since G_y has no perfect matching, we have $|S| + |T| < n/2$ by Kőnig–Egerváry theorem.
- Thus it suffices to show $\varepsilon > 0$. Because (S, T) is a vertex cover of G_y, any edge $e = ij \in E$ with $i \notin S$, $j \notin T$ is not tight, i.e., $y_i + y_j - w_e > 0$. □

Hungarian method

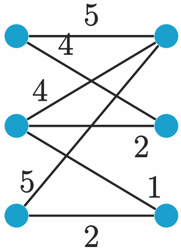
```
Set $y_i := \max_{e\in\delta(i)} w_e$ $(i \in V^+)$, $y_j := 0$ $(j \in V^-)$.        // $y$ is a trivial $w$-cover
while True :
    Find a maximum (unweighted) matching $M$ of $G_y$.        // $O(mn)$ time
    if $M$ is perfect :
        return $M$        // By optimality condition, $M$ is maximum-weight
    else        // $w$-cover update
        Let $X$ be the set of vertices reachable from $U^+$ in $D_M(y)$.
                // $(S,T) = (V^+ \setminus X, V^- \cap X)$ is a minimum vertex cover of $G_y$
        $\varepsilon := \min\{y_i + y_j - w_{ij} : i \in V^+ \cap X, j \in V^- \setminus X\}$
        $y_i \leftarrow y_i - \varepsilon$ $(i \in V^+ \cap X)$, $y_j \leftarrow y_j + \varepsilon$ $(j \in V^- \cap X)$
```

Example



G

Example



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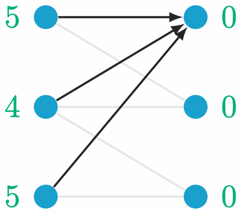
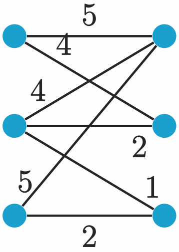


$y(V) = 14,$

Example



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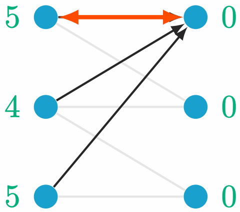

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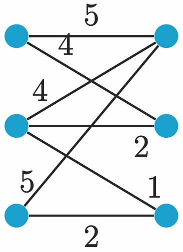

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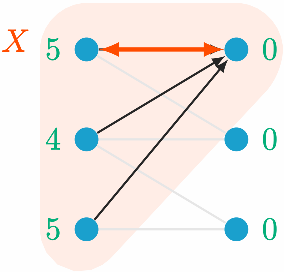

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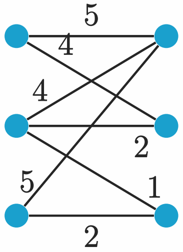

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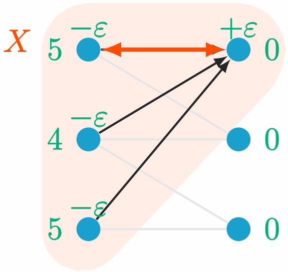
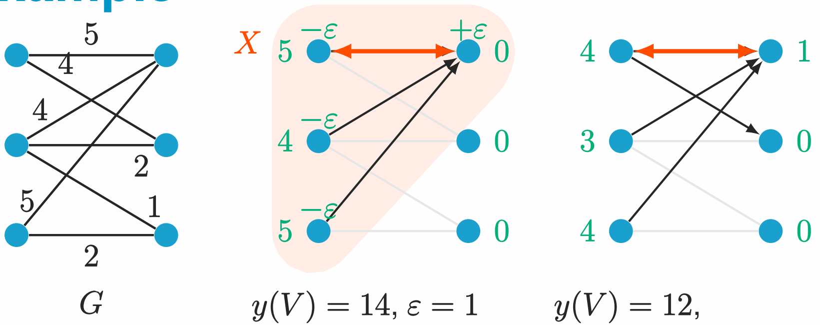


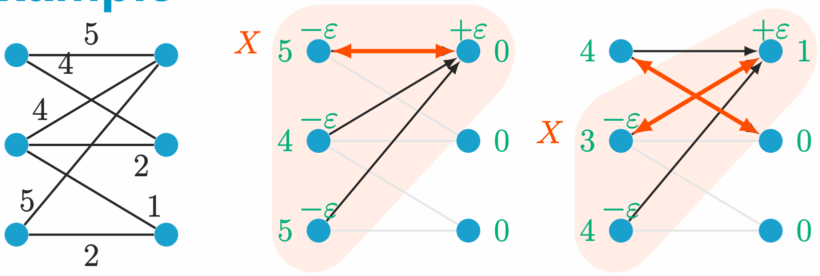
$y(V) = 14$, $\varepsilon = 1$

Example



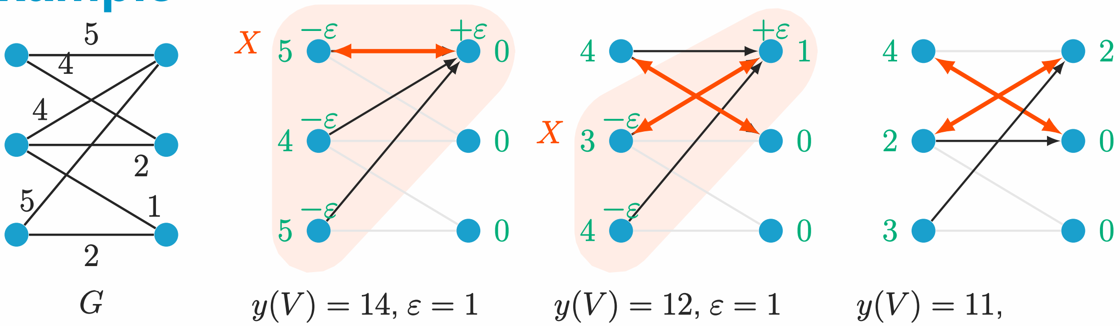
$y(V) = 14$, $\varepsilon = 1$ $y(V) = 12$,

Example



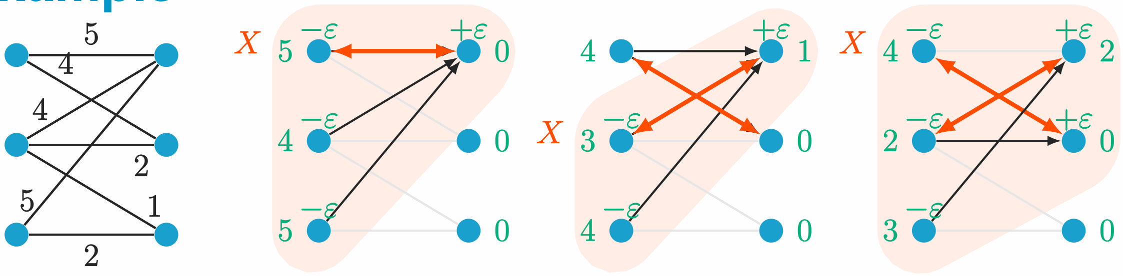
G $\quad$ $y(V) = 14$, $\varepsilon = 1$ $\quad$ $y(V) = 12$, $\varepsilon = 1$

Example



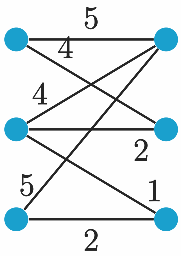
5
4
4
2
5
1
2
G
X
$-\varepsilon$
$+\varepsilon$
5
0
$-\varepsilon$
4
0
$-\varepsilon$
5
0
$y(V) = 14$, $\varepsilon = 1$
$+\varepsilon$
4
1
X
$-\varepsilon$
3
0
$-\varepsilon$
4
0
$y(V) = 12$, $\varepsilon = 1$
4
2
2
0
3
0
$y(V) = 11$,

Example



G $\quad$ $y(V) = 14,\ \varepsilon = 1$ $\quad$ $y(V) = 12,\ \varepsilon = 1$ $\quad$ $y(V) = 11,\ \varepsilon = 1$

Example



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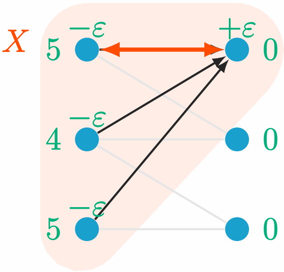
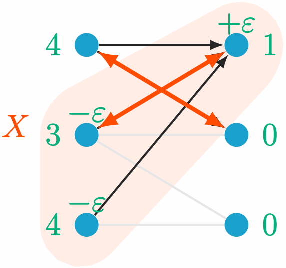
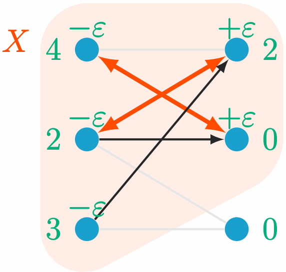


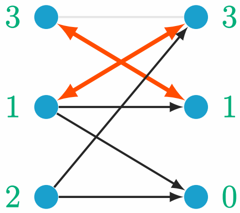
$y(V) = 14$, $\varepsilon = 1$



$y(V) = 12$, $\varepsilon = 1$

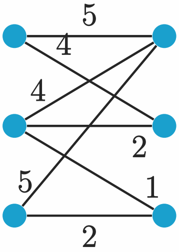


$y(V) = 11$, $\varepsilon = 1$



$y(V) = 10$

Example



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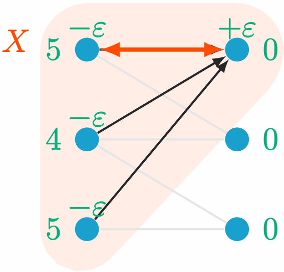

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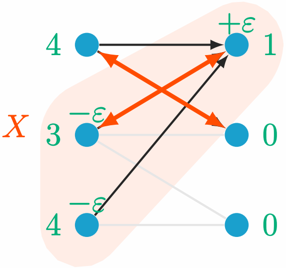

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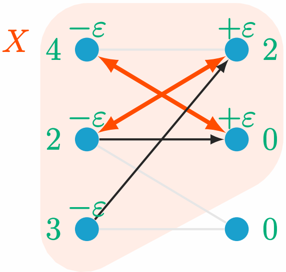

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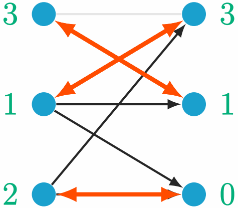

$y(V) = 10 = w(M)$ [END]

Analysis of the Hungarian method

Theorem

The Hungarian method finds a maximum-weight perfect matching and a minimum w-cover in $O(mn^3)$ time. $(m = |E|, n = |V|)$

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Therefore, the total number of iterations is $O(n^2)$. $\square$

Summary

- Basic LP-based ideas in combinatorial optimization (integral polyhedra and totally unimodular matrices)
- Using LP integrality, we proved the min-max theorem for bipartite matching (Kőnig–Egerváry theorem)
- Combinatorial algorithms that do not use LP
  - augmenting-path algorithm for unweighted bipartite matching
  - Hungarian method for weighted bipartite perfect matching