Polyhedral Combinatorics

Introduction to Computational Mathematics

Instructor: **Tasuku Soma**

July 16, 2026

Outline

1. What is combinatorial optimization?

- What is combinatorial optimization?
- What does it mean to “solve” a problem?
- What is polynomial time?

2. Review of linear programming

- Primal and dual problems
- Complementary slackness
- Polyhedra

3. Integral polyhedra and totally unimodular matrices

- Totally unimodular matrices
- Integral polyhedra
- Integral polyhedra and integrality of LP

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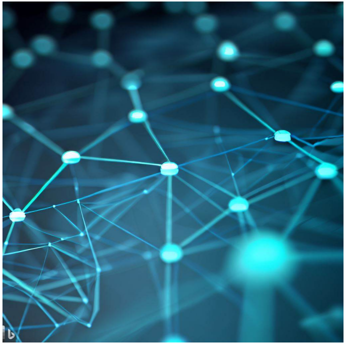
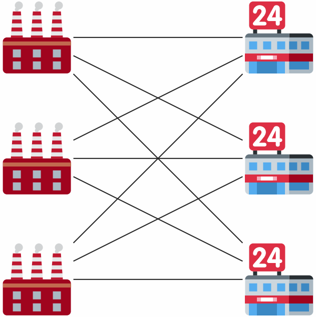
What is combinatorial optimization?

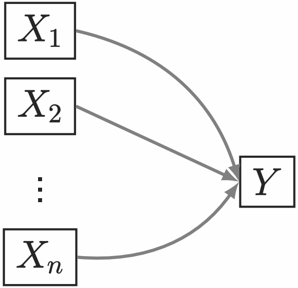
A methodology for finding the best solution among **discrete** objects **efficiently**.



Networks



Matchings



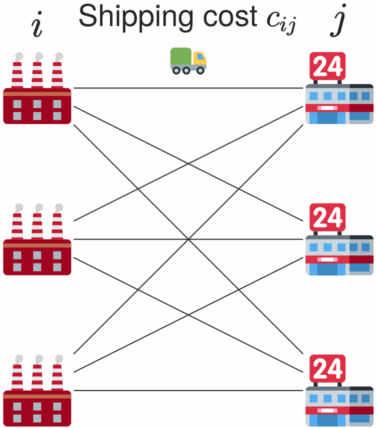
Machine learning and statistics

Example: Assignment problem

Assignment problem

Input Costs $c_{ij} \in \mathbb{R}$ $(1 \leq i, j \leq n)$

Output A matching with minimum total cost



matching: a set of edges s.t. every vertex is incident to exactly one edge.

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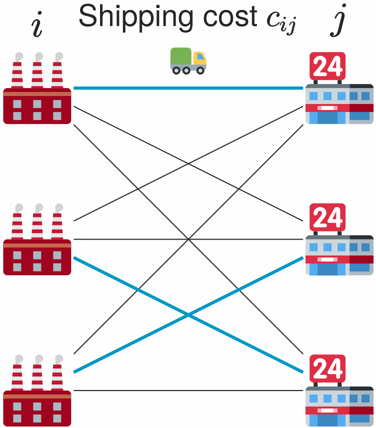
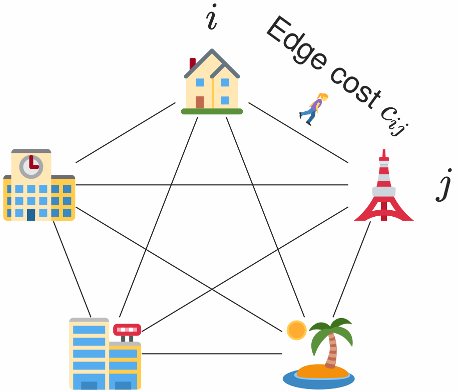


Total cost $= c_{11} + c_{23} + c_{32}$.

Example: Traveling salesperson problem (TSP)

Traveling salesperson problem

Input Edge costs $c_{ij} \in \mathbb{R}$ $(1 \leq i, j \leq n)$
Output A minimum-cost cycle visiting each vertex exactly once



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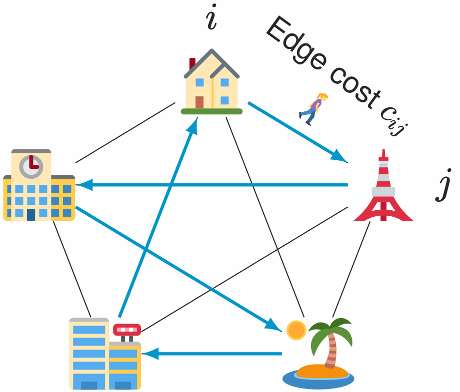
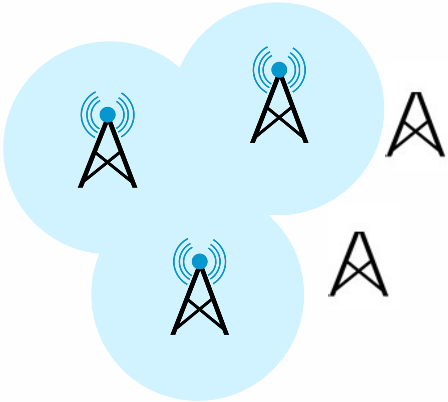


Example: Maximum coverage problem

Maximum coverage problem

Input V: candidate base-station locations, $k \in \mathbb{Z}_+$
Output $S \subseteq V$ with $|S| \leq k$ that maximizes the covered area



A naive approach: Brute force search

Examine all solutions and choose the best one!

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For TSP

Suppose we can examine 10^{17} solutions per second.

# of vertices n	Computation time $\approx n!$
10	3.6×10^{-11} sec
20	24 sec
30	31.68 million years

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This is impractical, so we need to develop more **efficient algorithms**.

What does “efficient” mean? 🤔

Time complexity of algorithms

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The time complexity of an algorithm is the **maximum** number of **elementary operations** (arithmetic operations, comparisons, bit operations, ...) performed by it.

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n: size of input

Polynomial time At most a polynomial number of elementary operations in n

Examples: $O(n)$, $O(n^2)$, $O(n \log n)$, and $O(n^{10000})$ time

Exponential time $O(2^n)$ or more elementary operations

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"Maximum": The time needed for the **hardest** input for the algorithm (worst-case time complexity).

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Linear programming (LP)

Any LP can be written in the following standard form:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (i = 1, \ldots, m) \\
& x_j \ge 0 \qquad\qquad\quad\;\; (j = 1, \ldots, n)
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$$\begin{aligned} \text{maximize} \quad & c^\top x \\ \text{subject to} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

- A: $m \times n$ matrix
- b: m-dim vector
- c: n-dim vector
- x: n-dim decision vector

Strong duality

Primal (P)

$$\begin{aligned} \text{maximize} \quad & c^\top x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual (D)

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Lemma (Weak duality)

For any feasible solution x to the primal problem (P) and any feasible solution y to the dual problem (D), $c^\top x \leq b^\top y$.

(pf) $c^\top x \leq (A^\top y)^\top x = y^\top (Ax) \leq y^\top b \quad \square$ $\square$

Strong duality

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Dual (D)

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Theorem (Strong duality)

If (P) and (D) are both feasible, they have optimal solutions x^* and y^* such that $c^\top x^* = b^\top y^*$.

Complementary slackness

Lemma

Let x, y be feasible solutions to the primal problem (P) and the dual problem (D), respectively. Then, x and y are optimal,
$\iff (A^\top y - c)^\top x = 0$ *and* $(b - Ax)^\top y = 0$

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(pf) For feasible solutions (x, y),

$$c^\top x \le (A^\top y)^\top x = y^\top (Ax) \le y^\top b.$$

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If (x, y) is optimal, strong duality yields $c^\top x = y^\top b$, so every inequality above is an equality. Therefore,

$$c^\top x = (A^\top y)^\top x, \quad y^\top (Ax) = y^\top b.$$

The converse follows similarly. □

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Theorem (Complementary slackness)

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$$\iff \begin{cases} (A^\top y - c)_j x_j = 0 & (j = 1, \ldots, n) \\ (b - Ax)_i y_i = 0 & (i = 1, \ldots, m) \end{cases}$$

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(pf) By the lemma, (x, y) optimal $\iff$

$$(A^\top y - c)^\top x = 0, \quad (b - Ax)^\top y = 0$$

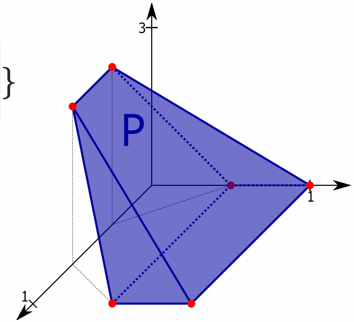
Since $A^\top y - c \ge 0$, $x \ge 0$, $b - Ax \ge 0$, and $y \ge 0$, every componentwise product is nonnegative. Thus, they must all be zero. □

Polyhedra and extreme points

The feasible region of an LP,

$$P = \{x \in \mathbb{R}^n : Ax \leq b,\, x \geq 0\} = \{x \in \mathbb{R}^n : \underbrace{\begin{bmatrix} A \\ -I \end{bmatrix}}_{\tilde{A}} x \leq \underbrace{\begin{bmatrix} b \\ 0 \end{bmatrix}}_{\tilde{b}} \}$$

is a convex set called a **polyhedron**.



Lemma

For every extreme point x of P, there exists a row subset S of size n s.t.

- *The submatrix $\tilde{A}_S$ of $\tilde{A}$ indexed by S is nonsingular.*
- *x is the unique solution to $(\tilde{A}_S)x = \tilde{b}_S$, where $\tilde{b}_S$ is the subvector of $\tilde{b}$ indexed by S.*

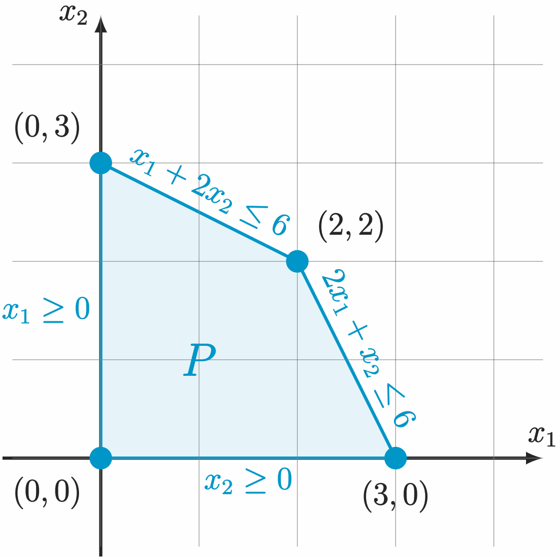
Example

Let P

$$\begin{cases} x_1 + 2x_2 \leq 6 \\ 2x_1 + x_2 \leq 6 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$

be a polyhedron in $\mathbb{R}^2$.

$$\tilde{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 6 \\ 6 \\ 0 \\ 0 \end{bmatrix}$$



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Solving combinatorial optimization with LP?

1. Formulate a combinatorial optimization problem as a **0–1 integer program (IP)**.

2. Solve the LP obtained by dropping the 0–1 constraints of the IP.

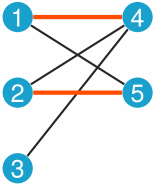
3. If we got an $\{0,1\}$-LP solution, then it is also an optimum of the original IP, so the problem is solved (lucky!).

A sufficient condition for this to work: **total unimodularity**

Example: Bipartite matching

Weighted bipartite matching problem

Input $G = (V; E)$: a bipartite graph,
edge weights w_e $(e \in E)$
Output A maximum-weight matching M of G



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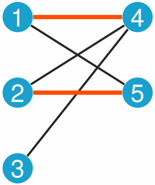

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IP formulation

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} w_e x_e \\ \text{subject to} \quad & \sum_{e \in \delta(i)} x_e \leq 1 \quad (i \in V) \\ & x_e \in \{0, 1\} \quad (e \in E) \end{aligned}$$

$\delta(i)$: the set of edges incident to vertex i

$$\begin{aligned} \max \quad & w^\top x \\ \text{s.t.} \quad & x_{14} + x_{15} && \leq 1 \\ & x_{24} + x_{25} && \leq 1 \\ & x_{34} && \leq 1 \\ & x_{14} + x_{24} + x_{34} && \leq 1 \\ & x_{15} + x_{25} && \leq 1 \\ & x_{14}, x_{15}, x_{24}, x_{25}, x_{34} && \in \{0, 1\} \end{aligned}$$

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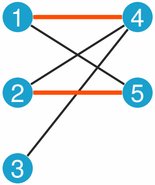

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Goal: Total unimodularity of the coefficient matrix ensures an LP optimal solution with **0–1** entries!

Totally unimodular matrices

Definition (Totally unimodular matrix)

A matrix A is **totally unimodular** $\overset{\text{def}}{\iff}$ every subdeterminant of A is either 0 or ± 1.

Example

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

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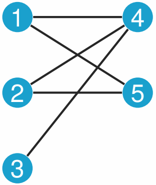
Lemma

The coefficient matrix A of the LP for bipartite matching is totally unimodular.



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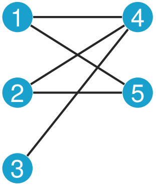
The coefficient matrix A of the LP for bipartite matching is totally unimodular.

(pf) We prove by induction on k that every $k \times k$ subdeterminant belongs to $\{0, \pm 1\}$.
For $k = 1$, this is trivial because every entry of A is 0 or 1.
Let $k > 1$, and consider a $k \times k$ submatrix A'.



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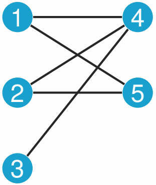

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- If A' has a zero column, then $\det A' = 0$.
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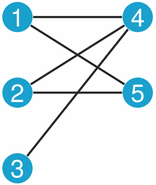

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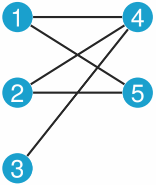

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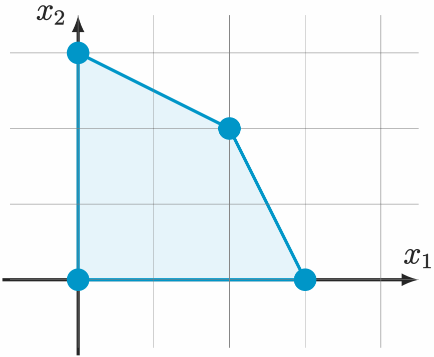
- Otherwise, every column has two 1s. Let v assign $+1$ to all left vertices and -1 to all right vertices. Then $vA' = 0$, so A' is singular and $\det A' = 0$. $\square$

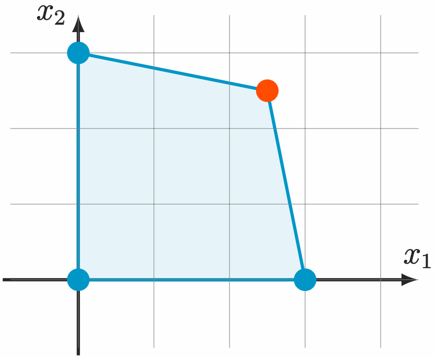
Integral polyhedra

Definition (Integral polyhedron)

A polyhedron is **integral** $\overset{\text{def}}{\iff}$ Each extreme point is an integral vector.



Integral polyhedron



Non-integral polyhedron

TU matrices and integral polyhedra

Theorem

For a totally unimodular matrix A and an integral vector b, the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$ is integral.

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(pf) Let x be any extreme point of P. There are a nonsingular submatrix A' and a subvector b', obtained by selecting n rows from $(\tilde{A}, \tilde{b})$, such that $x = (A')^{-1}b'$.

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By Cramer's rule,

$$(A')^{-1}_{ij} = \frac{\Delta_{j,i}A'}{\det A'} \in \{0, \pm 1\}$$

Here, $\Delta_{j,i}A'$ is the (j,i)-cofactor of A'.

Since b' is integral, $(A')^{-1}b'$ is also integral. $\square$

Total unimodularity and integrality of LP

Theorem

Let A be a totally unimodular matrix and b an integral vector. If (P) has an optimal solution, then it has an optimal solution that is an ***integral vector****.*

Primal (P)

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

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$$\begin{array}{ll} \text{max} & c^\top x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array}$$

(pf) By the preceding theorem, the feasible region P of (P) is an integral polyhedron. Since (P) has an optimal solution, it has an optimal solution that is an extreme point of P. By the definition of an integral polyhedron, this solution is an integral vector. □